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Letter to the Editor

Remarks on "Differential equation of Appell polynomials..."

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Abstract

We show that a recent result of He and Ricci (J. Comp. Appl. Math. 139 (2002) 231) on differential equations for Appell polynomials is valid for all polynomial sequences.

1. Introduction

He and Ricci [1, Theorem 2.1] showed that if $\{P_n(x)\}$ is a sequence of Appell polynomials then they satisfy

$$xy' - ny + \sum_{k=0}^{n-1} b_k y^{(k+1)} = 0, \tag{1.1}$$

where $\{P_n(x)\}$ have the explicit form

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k} \tag{1.2}$$

and the coefficients b_j are generated recursively from

$$a_{k+1} = \sum_{j=0}^{k} \binom{k}{j} a_j b_{k-j}. \tag{1.3}$$

In general it is desirable to find differential equations of fixed order for a sequence of polynomials $\{P_n(x)\}$, with P_n of degree n. So the advantage of finding differential equations like (1.1) is not clear. It turns out however that equations of the type (1.1) always exist, see (2.3). Moreover if a

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polynomial of degree m satisfies Eq. (2.3) then $m = n$ and the polynomial solution is a constant multiple of P_n. This latter fact was not proved in the case of Appell polynomials in [1].

The idea is to first observe that $xP_n'(x) - nP_n(x)$ is a polynomial of degree $n-1$, so by the division algorithm it is a constant multiple of $P_n'(x)$ plus a remainder. One can continue this process by the division algorithm. In Section 2 we formalize this procedure.

We also prove a similar result for any linear operator T, with $Tx^n = c_n x^{n-1}$, with $c_0 = 0$, $c_n \neq 0$ for $n > 0$. In particular, this contains the q-difference operator D_q, $(D_q f)(x) = [f(x) - f(qx)]/[(1-q)x]$.

The proof given by He and Ricci used raising and lowering operator techniques which are not available for general polynomials. The proof given here does not use ladder operators, although lowering operators always exist, see the Sheffer classification in [2].

2. A generalization

In this section we prove the following generalization of Theorem 2.1 of [1].

Theorem 2.1. *Let*

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_{n,k} x^{n-k}, \tag{2.1}$$

and define a sequence $\{b_{n,k}\colon 0 \leqslant k < n\}$ *by*

$$a_{n,k+1} = \sum_{j=0}^{k} \binom{k}{j} a_{n,j} b_{n,k-j}, \quad k = 0, 1, \ldots, n-1. \tag{2.2}$$

Then $P_n(x)$ *satisfies*

$$\sum_{k=0}^{n-1} \frac{b_{n,k}}{k!} y^{(k+1)} + xy' - ny = 0. \tag{2.3}$$

Furthermore if $\phi(x)$ *is a polynomials solution to* (2.3) *of degree* m *then* $m = n$ *and* ϕ *is a constant multiple of* P_n.

Proof. Clearly

$$\begin{aligned} &nP_n(x) - xP_n'(x) \\ &\quad = \sum_{s=0}^{n} \binom{n}{s} a_{n,s}[n-(n-s)]x^{n-s} \quad = n \sum_{s=0}^{n-1} \binom{n-1}{s} a_{n,s+1} x^{n-s-1}. \end{aligned} \tag{2.4}$$

On the other hand we have

$$\sum_{j=0}^{n-1}\frac{b_{n,j}}{j!}\frac{\mathrm{d}^{j+1}}{\mathrm{d}x^{j+1}}P_n(x)$$

$$=\sum_{j=0}^{n-1}\frac{b_{n,j}}{j!}\sum_{k=0}^{n-j-1}\frac{n!}{k!}a_{n,k}\frac{x^{n-k-j-1}}{(n-k-j-1)!}\quad=\sum_{s=0}^{n-1}\frac{x^{n-s-1}}{(n-s-1)!}\sum_{k=0}^{s}\frac{b_{n,s-k}}{(s-k)!}\frac{n!}{k!}a_{n,k}$$

$$=\sum_{s=0}^{n-1}\frac{n!}{s!\,(n-s-1)!}x^{n-s-1}a_{n,s+1}=\sum_{r=0}^{n-1}\binom{n}{r}a_{n,r}(r-n+n)x^{n-r}=nP_n(x)-xP_n'(x),$$

where we used (2.2) in the last step. This proves (2.3). Let $\phi(x)=\sum_{s=0}^{n}c_sx^{n-s}$. Substitute for ϕ in (2.3) to see that c_0 is arbitrary and the remaining c's are uniquely determined in terms of c_0.

Now consider a linear operator T with $Tx^n=c_nx^{n-1}$, where $c_0=0$ and $c_n\neq 0$ for $n>0$. Let c stand for the sequence $\{c_n\}$. Define the (generalized) binomial coefficient with respect to $\{c\}$ by

$$\binom{n}{0}_c=1,\qquad \binom{n}{k}_c=\frac{c_n\cdots c_{n-k+1}}{c_1\cdots c_k}.$$

With this notation we let

$$R_n(x)=\sum_{k=0}^{n}\binom{n}{k}_c a_{n,k}x^{n-k},$$

and define $\{b_{n,k}\colon 0\leqslant k<n\}$ via

$$a_{n,k+1}=\sum_{j=0}^{k}\binom{k}{j}_c a_{n,j}b_{n,k-j},\qquad k=0,1,\ldots,n-1.$$

Then $R_n(x)$ satisfies

$$b_{n,0}Ty+\sum_{k=1}^{n-1}\frac{b_{n,k}}{c_1\cdots c_k}T^{(k+1)}y+xTy-c_ny=0.$$

Furthermore if $\phi(x)$ is a polynomials solution to (2.8) of degree m then $m=n$ and ϕ is a constant multiple of R_n. The proof is identical to our proof of Theorem 2.1 and will be omitted.

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[2] E.D. Rainville, Special Functions, Chelsea, The Bronx, 1971.